Week 8 Worksheet - Taylor Polynomials/Series (Computing and substitution)

For your answers, you could use either $\cdots$ or the $\sum$ notation. In this solution, both of them are provided, but you don't need to have both on your hw or quiz.

The Taylor polynomial of a function $y = f(x)$ of degree n around the point $x = a$ is

$$T_n^a f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots\cdots \frac{f^{(n)}(a)}{n!}(x-a)^n$$

1. Compute the degree two Taylor polynomial of the function $f(x) = e^{\tan x}$ around 0. Use this to estimate $e^{\tan(0.1)}$.

   Solution: $n = 2, a = 0.\quad f(0) = 1$

   $f'(x) = e^{\tan x}\sec^2 x \Rightarrow f'(0) = 1$

   $f''(x) = e^{\tan x}\sec^2 x \cdot \sec^2 x + e^{\tan x} \cdot 2\sec x \cdot \sec x \tan x \Rightarrow f''(0) = 1 + 0 = 1$

   So $T_2^0 f(x) = 1 + x + \frac{1}{2}x^2$.

   $e^{\tan(0.1)} \approx 1 + 0.1 + \frac{1}{2}0.1^2 = 1.105$.

2. *(2015 fall exam)* Find $T_1^2 x^3$.

   Solution: $n = 1,\ a = 2.\ f(2) = 8.$

   $f'(x) = 3x^2 \Rightarrow f'(2) = 12$

   So $T_1^2 x^3 = 8 + 12(x-2)$.

3. *(2015 fall exam)* Find $T_2 \cos(\sin x)$.

   Solution: $n = 2,\ a = 0.\quad f(0) = \cos(0) = 1.$

   $f'(x) = -\sin(\sin x)\cdot\cos x \Rightarrow f'(0) = 0$

   $f''(x) = -\cos(\sin x)\cdot\cos x\cdot\cos x - \sin(\sin x)\cdot(-\sin x) \Rightarrow f'(0) = -1$

   So $T_2\cos(\sin x) = 1 - \frac{1}{2}x^2$.

The following problems are using substitution and the 5 formulas on the textbook page 78.

4. Find the Taylor series $T_\infty f(x)$ for the function $f(x) = \sin(2x)$.

   Solution: Since $\sin u = u - \frac{u^3}{3!} + \frac{u^5}{5!} - \frac{u^7}{7!} + \cdots = \sum\limits_{n=0}^{\infty}(-1)^n \frac{u^{2n+1}}{(2n+1)!}$,then take $u = 2x$, we have

   $\sin(2x) = 2x - \frac{2^3x^3}{3!} + \frac{2^5x^5}{5!} - \frac{2^7x^7}{7!} + \cdots = \sum\limits_{n=0}^{\infty}(-1)^n\frac{2^{2n+1}x^{2n+1}}{(2n+1)!}$.

5. Find the Taylor series $T_\infty f(x)$ for the function $f(x) = \frac{1}{1+\pi t}$. (What about $\frac{t}{1+\pi t}$?)

   Solution: Since $\frac{1}{1-u} = 1 + u + u^2 + u^3 + \cdots = \sum\limits_{n=0}^{\infty} u^n$, then take $u = -\pi t$, we have

   $\frac{1}{1+\pi t} = 1 - \pi t + \pi^2 t^2 - \pi^3 t^3 + \cdots = \sum\limits_{n=0}^{\infty}(-\pi t)^n = \sum\limits_{n=0}^{\infty}(-\pi)^n t^n$.

   $\frac{t}{1+\pi t} = t\left(1 - \pi t + \pi^2t^2 - \pi^3t^3 + \cdots\right) = \sum\limits_{n=0}^{\infty} t(-\pi t)^n = \sum\limits_{n=0}^{\infty}(-\pi)^n t^{n+1}$.

6. Find the Taylor series $T_\infty f(x)$ for the function

(a) $f(x) = \ln(2+2t)$.

(b) $f(x) = \ln\sqrt{\frac{1-t}{1+t}}$.

Solution:

(a) $f(x) = \ln[2(1+t)] = \ln 2 + \ln(1+t)$.

Since $\ln(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{t^n}{n}$, then

$f(x) = \ln 2 + t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots = \ln 2 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{t^n}{n}$.

(b) $f(x) = \frac{1}{2}\ln\frac{1-t}{1+t} = \frac{1}{2}\ln(1-t) - \frac{1}{2}\ln(1+t)$

$$f(x) = \frac{1}{2}\left[-t - \frac{(-t)^2}{2} + \frac{(-t)^3}{3} - \frac{(-t)^4}{4} + \cdots\right] - \frac{1}{2}\left[t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots\right]$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(-1)^n t^n}{n} + \frac{1}{2}\sum_{n=1}^{\infty} (-1)^{n+1} \frac{t^n}{n}$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} -\frac{t^n}{n} + \frac{1}{2}\sum_{n=1}^{\infty} (-1)^{n+1} \frac{t^n}{n}$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} \left[(-1)^{n+1} - 1\right] \frac{t^n}{n}.$$